THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2015–2016)
Introduction to Topology
Exercise 8 Quotient

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $X = \{ (x,0) : x \in \mathbb{R} \} \cup \{ (x,1) : x \in \mathbb{R} \} \subset \mathbb{R}^2$, i.e., $X = \mathbb{R} \coprod \mathbb{R}$. Define an equivalence relation on X by identifying $(0,0)$ and $(0,1)$. Rigorously, this means $(s_1,t_1) \sim (s_2,t_2)$ iff $(s_1,t_1) = (s_2,t_2)$ or $(s_1,t_1) = (0,0)$ while $(s_2,t_2) = (0,1)$ or vice versa. Show that $X/{\sim}$ is homeomorphic to the two axes in $\mathbb{R}^2$.

2. Let $X = \left\{ (s,t) \in \mathbb{R}^2 : 0 \neq t \in \mathbb{Z} \right\}$ and $Y = \left\{ (s,t) \in \mathbb{R}^2 : 1/t \in \mathbb{Z} \right\}$ be given the standard induced topology. Define an equivalence relation on both X and Y by $(s_1,t_1) \sim (s_2,t_2)$ iff $(s_1,t_1) = (s_2,t_2)$ or $s_1 = s_2 = 0$. That is points on the y-axis are identified to one point. Is it true that $X/{\sim}$ and $Y/{\sim}$ are homeomorphic?

3. Define an equivalence relation on $\mathbb{R}$ by identifying n with $1/n$ for all $n \in \mathbb{Z}$.

   (a) Sketch a picture to represent the space $\mathbb{R}/{\sim}$.

   (b) Find a sequence $x_n \in \mathbb{R}$ such that $[x_n] \in \mathbb{R}/{\sim}$ converges but x_n does not.

   (c) Can a sequence $x_n \in \mathbb{R}$ converge but $[x_n] \in \mathbb{R}/{\sim}$ does not?

4. Let $X/{\sim}$ be a quotient space obtained from X and $Y \subset X$.

   (a) Show that there is a natural way to induce an equivalence relation on Y; and thus a quotient space $Y/{\sim}$.

   (b) Let $Y^* = \{ [x] \in (X/{\sim}) : [x] \cap Y \neq \emptyset \}$ be given the topology induced from $X/{\sim}$. Is Y^* homeomorphic to $Y/{\sim}$?

5. Let $X = \mathbb{R}/{\sim}$ where $s \sim t$ if $s, t \in \mathbb{Z}$; also let

$$Y = \bigcup_{n=1}^{\infty} \left\{ z \in \mathbb{C} : \left| z - \frac{1}{n} \right| = \frac{1}{n} \right\} \quad \text{and} \quad Z = \bigcup_{n=1}^{\infty} \{ z \in \mathbb{C} : |z - n| = n \}$$

   with induced topology of $\mathbb{C} = \mathbb{R}^2$. Are X, Y, Z homeomorphic to each other?

6. Let $A \subset X$. What can you say about $\mathrm{Int}(A)/{\sim}$ and $\mathrm{Int}(A/{\sim})$; $\mathrm{Cl}(A)/{\sim}$ and $\mathrm{Cl}(A/{\sim})$?

7. Let $\mathfrak{T}_q$ be the quotient topology on $X/{\sim}$ and $\mathfrak{T}'$ be any topology. Show that if the quotient map $q\colon\ (X,\mathfrak{T}) \to (X/{\sim},\mathfrak{T}')$ is continuous, then $\mathfrak{T}' \subset \mathfrak{T}_q$.

8. Let Z be any topological space. A mapping $f\colon (X/{\sim}, \mathfrak{T}_q) \to Z$ is continuous if and only if $f \circ q\colon (X, \mathfrak{T}) \to Z$ is continuous.

   If $\mathfrak{T}'$ is a topology on $X/{\sim}$ satisfying the same property, then $\mathfrak{T}' = \mathfrak{T}_q$.

9. On the disk $\mathbb{D} = \{\, z \in \mathbb{C} : \ |z| \le 1 \,\}$ with induced topology from standard $\mathbb{C}$, define an equivalence relation $\sim$ by $z_1 \sim z_2$ if $z_1 = z_2$ or $\overline{z}_1 = z_2$ with $|z_1| = 1 = |z_2|$. Show that $\mathbb{D}/{\sim}$ is homeomorphic to the sphere $\mathbb{S}^2$.

10. Define a mapping $p : \ \mathbb{D} \to \mathbb{S}^2$ in the following way. For $z = |z|\, e^{\mathbf{i}\theta} \in \mathbb{D}$, let

$$p(z) = (\, \sin(\pi\, |z|) \cos\theta, \sin(\pi\, |z|) \sin\theta, -\cos(\pi\, |z|)\,)\ .$$

   (a) Show that the quotient topology on $\mathbb{S}^2$ induced by p is the standard topology.

   (b) Define an equivalence relation $\sim$ on $\mathbb{D}$ such that $z_1 \sim z_2$ if $p(z_1) = p(z_2)$. What is the topological space $\mathbb{D}/{\sim}$?